

Breakdown of the Landau Fermi liquids: restriction on the degrees of freedom of quantum electrons

Yuehua Su^{1,*} and Hantao Lu^{2,†}

¹ *Department of Physics, Yantai University, Yantai 264005, P. R. China*

² *Center for Interdisciplinary Studies, Lanzhou University, Lanzhou 730000, P. R. China*

One challenge in the modern field of condensed matter is to understand the wide-ranging unconventional electronic physics beyond the paradigm of the Landau Fermi-liquid theory. In this article, we present a new perspective that most unconventional electronic physics stem from restriction on some degrees of freedom of the Landau Fermi liquids. The restriction on the degrees of freedom can be regarded as restriction on the fiber-bundle space of the electron quantum wavefunction or quantum field with both the *base* spatial-temporal degrees of freedom or the *fiber* internal ones involved. The restriction can be taken on from external constraints or in interaction-driven processes through mechanisms such as (i) reduction of the symmetries, (ii) emergence of new symmetries, and (iii) emergence of the nontrivial topology of the electron fiber-bundle space. Examples are extensively investigated from this point of view. Our perspective provides basic routes to study the breakdown of the Landau Fermi liquids. It can also serve as a guiding principle in search of novel electronic systems and devices.

I. INTRODUCTION

In the 1950s, L. P. Landau established a phenomenological theory for the low-energy physics of the Fermi particles, which is greatly powerful and triumphal well-known as the Landau Fermi-liquid theory^{1–3}. In the Landau Fermi-liquid theory, some principle hypotheses are assumed. One is the Pauli exclusion principle due to the Fermi-Dirac statistics. This restricts extremely the phase space of the low-energy Fermi particles near the Fermi energy. The other one is the adiabatic and analytic principle. When weak interactions *turn on* adiabatically in the non-interacting Fermi gas, the transition from the Fermi gas to the weakly interacting Fermi liquid evolves adiabatically and analytically. The free Fermi particles in the Fermi gas are one-to-one correspondence to the *quasiparticles* in the Fermi liquid. Landau then assumed that the low-energy free energy is a functional of the quasiparticle occupation numbers, i.e.,

$$F[n] = E[n] - TS[n], \quad (1)$$

where the energy $E[n]$ and the entropy $S[n]$ follow

$$E[n] = E_0 + \sum_{\mathbf{k}\sigma} \varepsilon_{\mathbf{k}\sigma}^{(0)} \delta n_{\mathbf{k}\sigma} + \frac{1}{2} \sum_{\mathbf{k}\sigma; \mathbf{k}'\sigma'} f_{\mathbf{k}\sigma, \mathbf{k}'\sigma'} \delta n_{\mathbf{k}\sigma} \delta n_{\mathbf{k}'\sigma'},$$

$$S[n] = - \sum_{\mathbf{k}\sigma} (n_{\mathbf{k}\sigma} \ln n_{\mathbf{k}\sigma} + (1 - n_{\mathbf{k}\sigma}) \ln(1 - n_{\mathbf{k}\sigma})).$$

$\varepsilon_{\mathbf{k}\sigma} = \frac{\delta E[n]}{\delta n_{\mathbf{k}\sigma}}$ is the energy of the quasiparticle with momentum \mathbf{k} and spin σ and $n_{\mathbf{k}\sigma} = \frac{1}{e^{(\varepsilon_{\mathbf{k}\sigma} - \mu_F)/T} + 1}$ is the Fermi-Dirac function.

The Landau Fermi-liquid theory can account for many universal properties of the low-energy Fermi systems⁴, such as the well-defined Fermi surface, the linear-T specific heat, the Pauli spin susceptibility, the emergence of the zero sounds, etc. The greatly successful Landau Fermi-liquid theory has been intensively re-established

from the quantum field theory⁵ and the renormalization group theory⁶. The latter shows the stability of the Landau Fermi liquids in two- and three-dimensional (2D and 3D) spatial spaces, where the Landau interactions which involve only forward scatterings are marginal.

In the last several decades, much attention in the condensed matter field has focused on the unconventional electrons in breakdown of the Landau Fermi-liquid theory^{7,8}, such as one-dimensional (1D) Luttinger liquids⁹, disordered electrons with Anderson localization¹⁰, heavy fermion superconductors^{8,11}, high-Tc cuprate superconductors^{12–15}, 2D electrons with quantum Hall effects, Fe-based superconductors^{16,17}, etc. The unconventional electronic physics include (i) universal 1D Luttinger liquids, (ii) electronic Anderson localization and metal-insulator transition, (iii) integer and fractional quantum Hall effects, (iv) unconventional superconductivity and its novel mother normal states, (v) quantum phase transitions and quantum criticality, etc.

The special theoretical formalisms have been well-established for the 1D Luttinger liquids¹⁸ and the quantum Hall effects^{19,20}. The Landau's theory of spontaneous symmetry breaking²¹ and the Wilson's renormalization group theory²² can account well the classical continuous phase transitions and the corresponding critical phenomena. However, there are still many unconventional electronic physics beyond these well-established theories. The unconventional superconductivity, the various novel normal states, the quantum phase transitions and criticality etc. are still far beyond our knowledge.

In this article, we will present a new perspective for the emergence of the unconventional electronic physics with breakdown of the Landau Fermi liquids. Our perspective is that most of the unconventional electronic physics stem from restriction on some degrees of freedom of the electrons in the Landau Fermi liquids. The restriction on the degrees of freedom can be regarded equivalent to the restriction on the fiber-bundle space of the electron quantum wavefunction or quantum field. Thus the de-

degrees of freedom involve both the *base* spatial-temporal ones such as the position, the momentum and the time variable, and the *fiber* internal ones such as the spin, the charge, the phase, the local basis states, etc. The degrees of freedom of quantum electrons are deeply related to the symmetries and the fiber-bundle topology. Thus the restriction can be taken on, either from external constraints or in interaction-driven dynamical processes, from the mechanisms: (i) reduction of the symmetries; (ii) emergence of new symmetries; and (iii) emergence of the non-trivial topology of the fiber-bundle space. Our perspective shows us how to study the breakdown of the Landau Fermi liquids, i.e., to focus our attention on which degrees of freedom of the electrons are restricted and how the restriction is taken on. Our perspective also guide us in search of novel electronic devices from manual control of the electrons with operations on the degrees of freedom.

This article is arranged as following. In Section II, we show how to denote the electronic quantum states. It leads the definition of the degrees of freedom with the symmetries of the Hamiltonian and the fiber-bundle topology of the quantum wavefunction or quantum field involved. The principle for the breakdown of the Landau Fermi liquids is also present in this section. In the following sections, we review extensively examples of the unconventional electronic physics beyond the Landau Fermi-liquid theory from our perspective. Section III focuses on restriction on the spatial degree of freedom which can result in critical phenomena with anomalous dimensions and fractal physics, 1D universal Luttinger liquids, the impurity-induced Anderson localization. Section IV involves restriction on the fiber internal degrees of freedom, such as the spin, the charge, the phase or the local basis space, which can lead to quantum Hall effects, spin- and charge-density waves (SDW and CDW), Pomeranchuk nematicity, superconductivity, and Mott-Hubbard physics. Section V focuses on restriction from external degrees of freedom. The examples include the electrons coupled with Goldstone or gauge bosons, Anderson's hidden Fermi liquid, the electrons coupled with Bose unparticles, and the electrons coupled with local magnetic moments in Kondo effects and heavy fermion superconductors. Finally we give discussions in Section VI on the possible routes to the breakdown of the Landau Fermi liquids based on our perspective. Some proposals are also present in this section.

II. BREAKDOWN OF THE LANDAU FERMI LIQUIDS: BASIC IDEAS

A. Degrees of freedom and basic principle

The variables to describe classical electrons are the position \mathbf{r} , the momentum \mathbf{p} , the time t , the mass m and the charge e . For quantum electrons, the quantum states which are defined in a Hilbert space are basic vari-

ables. Any state in a Hilbert space can be expressed as $|\Psi(t)\rangle = \sum_{\alpha} \psi_{\alpha}(t) |\alpha\rangle$, where $\{|\alpha\rangle\}$ are the basis states, and $\psi_{\alpha}(t) \equiv \langle\alpha|\Psi(t)\rangle$ are the wavefunctions which can be quantized into the quantum field in second quantization formalisms. Following Dirac, the basis states $\{|\alpha\rangle\}$ are defined as eigenfunctions of a complete set of commuting observables with α being a set of quantum numbers²³. The complete set of commuting observables can be constructed from the symmetry group theory where the commuting observables are generators of the symmetry group²⁴. The quantum numbers α are indices of the irreducible representations of the symmetry group. The degrees of freedom of the quantum electrons are the variables which the generators of the symmetry group operate on.

The quantum states can also be described in a geometric language of fiber bundle⁷⁶. The quantum wavefunction or the quantum field $\psi_{\beta}(\mathbf{k}, t)$ can be regarded as a variable in a fiber-bundle space E , where the base space is the configuration spatial-temporal space $B = \{\mathbf{k}, t\}$, and the fiber space is a multi-component complex or quantum field operator space $F = \{\phi_{\beta}\}$ where β are the quantum numbers of the internal degrees of freedom. In the language of fiber bundle, the degrees of freedom of quantum electrons can be categorized in two spaces, the base spatial-temporal one and the fiber internal one. The symmetries which involves in the definition of the basis states can also be defined in these two different spaces. Moreover, there may be global nontrivial topological structure in the fiber-bundle space, which may take important roles in such as quantum Hall effects and topological insulators.

In the Landau Fermi liquids, there are spatial and temporal translational symmetries, spin SU(2) symmetry, charge U(1) symmetry. They make the momentum \mathbf{k} , the energy ω , the spin σ , the charge number n as well-defined quantum numbers^{1,4}. The degrees of freedom relevant to these symmetries are the variables of the spatial-temporal space and of the internal spin and phase spaces. The discrete inversion and time-reversal symmetries are also conserved in most Landau Fermi liquids. The spatial continuous or discrete rotational symmetry is also relevant to the Landau Fermi liquids, whose breaking will lead to Pomeranchuk Fermi-surface nematicity or electronic nematicity in Fe-based superconductors. The exchange anti-symmetry of the identical quantum electrons is one special symmetry defined for N -body states, which comes from Fermi-Dirac statistics and makes the Fermi surface as one essential relevant degree of freedom. The topology of the fiber-bundle space of the quantum wavefunction or quantum field is also relevant in novel quantum Hall states^{25,26} and topological insulators^{27,28}. Summary of the degrees of freedom is present in Table (I).

From the above definition of the electronic quantum states, we propose a new perspective that the breakdown of the Landau Fermi liquids can be made by restriction on some degrees of freedom of the quantum electrons.

TABLE I: Degree(s) of freedom (DOF) and quantum numbers (nos.) for the quantum electrons in the Landau Fermi liquids. The DOF are also classified in the base (B) or fiber (F) space in the geometric fiber-bundle language, and are defined on the 1-body or N-body state. The “addt.” and “mulpl.” are abbreviations which denote whether the quantum numbers are additive or multiplicative. We also include the Fermi surface (FS) as one relevant DOF for the N-body electrons. Topology of the fiber-bundle space is introduced for quantum Hall effects and topological insulators. Other abbreviations are defined as following: spc. - space, sym. - symmetry.

symmetry or topology	DOF	B/F spc.	quantum nos.	1-/N-body state
spatial translational sym.	\mathbf{r}	B	\mathbf{k} (addt.)	1
temporal translational sym.	t	B	ω (addt.)	1
spin SU(2) sym.	spin variable	F	σ (addt.)	1
charge U(1) sym.	phase θ	F	particle nos. n (addt.)	1
spatial rotational sym.	spatial angles	B	angular momentum (addt.)	1
inversion sym.	\mathbf{r}	B	+/- (mulpl.)	1
time-reversal sym.	t	B	+/- (mulpl.)	1
exchange anti-sym.	FS	F		N
topology	global structures of fiber bundle	B or F	topological nos.	N

Since the degrees of freedom of the quantum electrons are closely related to the symmetries and the fiber-bundle topology, the restriction can be done by modifying the symmetries or the fiber-bundle topology. The restriction can be taken on either from external constraints or in interaction-driven dynamical processes with the base spatial-temporal degrees of freedom or the fiber internal ones involved. The principal mechanisms for the breakdown of the Landau Fermi liquids can be as following: (i) reduction of the symmetries, such as the Landau’s spontaneous symmetry breaking for classical continuous phase transitions²¹; (ii) emergence of new symmetries, such as the conformal symmetry in 1D Luttinger liquids⁹ and the scaling symmetry at the critical point of continuous phase transitions²²; (iii) emergence of the nontrivial topology of the fiber-bundle space through stretching, compressing, screwing, or making “holes”, with examples such as the nontrivial topological structure in quantum Hall states^{25,26} and topological insulators^{27,28}.

B. Analyticity of single-particle Green’s function

The most central concept in the Landau Fermi liquids is the well-defined quasiparticles, which survive the interactions and are the principal low-energy excitations. In mathematics, the well-defined quasiparticles can be described by the single-particle Green’s function $G_\sigma(\mathbf{k}, \tau) = -\langle T_\tau c_{\mathbf{k}\sigma}(\tau) c_{\mathbf{k}\sigma}^\dagger(0) \rangle$, where τ is an imaginary time and the average is defined as $\langle A \rangle = \text{Tr}(\rho_H A)$ with ρ_H the density matrix. If the interactions are taken on adiabatically without any phase transition, a perturbation formalism can be established and the Dyson equation shows us that

$$G_\sigma(\mathbf{k}, i\omega_n) = \frac{1}{i\omega_n - \varepsilon_{\mathbf{k}\sigma}^{(0)} - \Sigma_\sigma(\mathbf{k}, i\omega_n)}, \quad (2)$$

where $\Sigma_\sigma(\mathbf{k}, i\omega_n)$ is the self-energy.

In the Landau Fermi-liquid states, the single-particle Green’s function can be separated into coherent and incoherent parts, $G_\sigma(\mathbf{k}, i\omega_n) = G_\sigma^{(coh)}(\mathbf{k}, i\omega_n) + G_\sigma^{(inc)}(\mathbf{k}, i\omega_n)$, where at low-energy the coherent part has a form:

$$G_\sigma^{(coh)}(\mathbf{k}, i\omega_n) = \frac{Z_{\mathbf{k}\sigma}}{i\omega_n - \varepsilon_{\mathbf{k}\sigma} - i\Gamma_\sigma(\mathbf{k}, i\omega_n)}. \quad (3)$$

Here $\varepsilon_{\mathbf{k}\sigma} = \varepsilon_{\mathbf{k}\sigma}^{(0)} + \Re\Sigma_\sigma(\mathbf{k}, \varepsilon_{\mathbf{k}\sigma})$ and $\Gamma_\sigma(\mathbf{k}, i\omega_n) = Z_{\mathbf{k}\sigma} \Im\Sigma_\sigma(\mathbf{k}, i\omega_n)$ with $Z_{\mathbf{k}\sigma} = \left(1 - \frac{\partial \Re\Sigma}{\partial \omega_n}\right)^{-1} \Big|_{i\omega_n = \varepsilon_{\mathbf{k}\sigma}}$. To separate out the coherent part of the Green’s function (3), an assumption is taken that the self-energy Σ is analytic near the Fermi energy.

One important property of $G_\sigma^{(coh)}(\mathbf{k}, z)$ is that in the complex z -plane, it has *one unique singular pseudo-pole* whose center locates at $\varepsilon_{\mathbf{k}\sigma} + i\Gamma_\sigma(\mathbf{k}, \varepsilon_{\mathbf{k}\sigma})$. In the case when the energy is much near the Fermi energy and $\Gamma_\sigma(\mathbf{k}, \varepsilon_{\mathbf{k}}) \rightarrow 0^+$, the pseudo-pole becomes a pure pole. $Z_{\mathbf{k}\sigma}$ is the residual weight of the singular pole. The emergence of the single singular pole is one manifestation of the well-defined quasiparticle, which has energy $\varepsilon_{\mathbf{k}\sigma}$, life time $1/\Gamma$ and coherent weight $Z_{\mathbf{k}\sigma}$. The single-pole structure in $G_\sigma^{(coh)}(\mathbf{k}, z)$ shows a peak structure in the spectrum function $A_\sigma(\mathbf{k}, \omega)$, which can be observed in angle-resolved photoemission spectroscopy.

There are some manifestations in the single-particle Green’s function when the Landau Fermi liquids are broken down.

1. Non analytic self-energy with $Z_{\mathbf{k}\sigma} \rightarrow 0^7$

If the Green’s function still follows the Dyson equation (2), but some singular scatterings can lead to non analytic self-energy, such as $\Sigma(\mathbf{k}_F, \omega) \sim \ln \frac{\omega_c}{\omega} + i|\omega|^{7,29}$ in the marginal Fermi liquid. Then the single-pole (or pseudo-pole) structure in the Green’s function is broken. Thus these singular scatterings can lead to breakdown of the Landau Fermi liquids.

This is one main route to breakdown of the Landau Fermi liquids from strong-coupling formalisms, such as in the study of the electrons coupled with spin fluctuations in high-Tc cuprate superconductors³⁰.

2. Single-pole singularity changed into multi-pole singularity⁹

Consider the superconducting (SC) mean-field state with Hamiltonian $H = \sum_{\mathbf{k}} \psi_{\mathbf{k}}^{\dagger} (\varepsilon_{\mathbf{k}} \tau_3 + \Delta_{\mathbf{k}} \tau_1) \psi_{\mathbf{k}}$, where τ_i are the Pauli matrices and $\psi_{\mathbf{k}}^{\dagger} = (c_{\mathbf{k}\uparrow}^{\dagger}, c_{-\mathbf{k}\downarrow})$ are the Nambu spinor operators. The single-particle Green's function follows

$$G_{\sigma}(\mathbf{k}, i\omega_n) = \frac{\alpha_{\mathbf{k}\sigma}}{i\omega_n - E_{\mathbf{k}}} + \frac{\beta_{\mathbf{k}\sigma}}{i\omega_n + E_{\mathbf{k}}}, \quad (4)$$

where $E_{\mathbf{k}} = \sqrt{\varepsilon_{\mathbf{k}}^2 + \Delta_{\mathbf{k}}^2}$. Compared to the single-pole singularity of the Green's function (3) in the Landau Fermi liquids, the Green's function in superconducting state has two separate poles, which locate symmetrically to the Fermi energy but with different coherent weights.

Similarly, the single-particle Green's function in the ordered SDW or CDW state also has two separated singular poles. The change of the single-pole singularity in the Landau Fermi liquids into the multi-pole singularity in the ordered state is one another route to breakdown of the Landau Fermi liquids. The recently proposed Yang-Rice-Zhang Green's function for the high-Tc cuprate superconductors also has separate singular poles³¹.

3. Single-pole singularity changed into branch-cut singularity

Suppose the Green's function has a form:

$$G_{\sigma}(\mathbf{k}, i\omega_n) = \frac{A}{(i\omega_n - \varepsilon_{\mathbf{k}\sigma})^{1-a}}, \quad (5)$$

where the exponent $a \neq 0$ is a real number. The analyticity of this Green's function is largely modified, since it now has branch-cut singularity.

The branch-cut singularity in the single-particle Green's function is one manifestation of the breakdown of the Landau Fermi liquids. It has been found extensively in 1D Luttinger liquids⁹ where the conformal symmetry leads to anomalous dimensions, in the Fermi liquids in fractal space³², in Anderson's Hidden Fermi liquid³³, and in the electrons coupled with unparticles with fractal dimension³⁴. We also propose that the branch-cut singularity can emerge in the electrons which are near a phase-transition critical point, where the self-similarity at the critical point can lead to anomalous dimension.

It should be noted that the single-particle Green's function can describe both the particle and the wave character of the quantum electrons which show particle-wave duality. The particle character involves the physics of separable particles which can be directly described by the quantum field theory. The wave character of the electrons are related to quantum interference in such as coherent propagating from one point to another in space. Both the path integral and the canonical quantum field theory can describe the coherent propagating

by the single-particle Green's function where the quantum interference from infinite paths is included.

III. RESTRICTION ON THE SPATIAL DEGREE OF FREEDOM

In this section, let us consider the restriction effects on the spatial degree of freedom of the quantum electrons. The self-similarity and the scaling symmetry at the critical point shows anomalous dimension and fractal physics. The Fermi liquids in reduced 1D spatial space have one fixed point, the universal Luttinger-liquids. The impurity-induced randomness can lead to Anderson localization and Anderson metal-insulator transition.

A. Self-similarity at critical point: anomalous dimension and fractal physics

Consider a thermal continuous phase transition. At the critical point, the electrons show scale invariance with correlation length divergent, $\xi \rightarrow \infty$. Near the critical phase transition, the thermal responses and the spatial correlations manifest singular power-law behaviors, which are rooted in the singularity of the correlation length near T_c , $\xi \sim |T - T_c|^{-\nu}$ ³⁵.

The most unusual mystery of critical phenomena is an emergence of the anomalous dimension η , which shows itself in the correlation function of the order parameter $\phi(\mathbf{r})$ as³⁶

$$\mathcal{G}(\mathbf{k}) \sim k^{-2+\eta}, \quad (6)$$

where $\mathcal{G}(\mathbf{k}) = \int d^d \mathbf{r} \langle \phi(\mathbf{r}) \phi(0) \rangle e^{i\mathbf{k} \cdot \mathbf{r}}$. The anomalous critical exponent in the correlation function stems from the anomalous dimension of the order parameter ϕ and comes from the novel correlation of the high- and low-energy modes. The finite anomalous dimension η violates the normal dimension analysis since the normal dimension of $\mathcal{G}(\mathbf{k})$ is -2 . In order to restore the correct normal dimension of \mathcal{G} , one can introduce a short-range length a_l so as $\mathcal{G}(\mathbf{k}) \sim a_l^{\eta} k^{-2+\eta}$ ³⁶. This shows the unusual role of the short-range fluctuations in critical phenomena.

The self-similarity and the finite anomalous dimension η lead us a new perspective for the critical phenomena: the critical phenomena are *fractal* physics. Near a critical phase transition with the emergent scaling symmetry, the electrons are restricted into an effective fractal spatial space which is renormalized by interactions and has an anomalous dimension η . The fractal nature of the critical phenomena has also been proposed previously^{37,38}.

We remark that the anomalous dimension and the fractal nature in critical phenomena are deeply related to an underlying novel non-linearity. Suppose one physical variable A has normal dimension d of volume ($V = L^d$) in dimension analysis. If it has scale dimension $d_s = d - \eta$ in critical region, then under scale transformation $l' = bl$, A follows $A'(L') = b^{-(d-\eta)} A(L)$. Thus we have

$A \sim L^{d-\eta} \sim V^{\frac{d-\eta}{d}}$, which shows that non-linear correlations dominate the critical behaviors of A . This is the non-linearity origin of the effective fractal dimension.

The physical effects of the fractal spatial space has been studied by Bares and Wen³², who studied the electrons in dimension $1 < d < 2$ where the Coulomb interaction has fractal power-law behaviors. They found non-Fermi liquid physics with branch-cut singular single-particle Green's function like (5).

The quantum phase transitions and the quantum criticality have been demonstrated in many strongly correlated electrons, where both the spatial and temporal relevant critical phenomena are still far beyond our understandings. We make a geometric proposal that these critical electrons are in an effective fractal spatial-temporal space, where the electrons show singular critical power-law behaviors with the critical exponents deeply related to the fractal dimension.

B. Dimension reduction: 1D Luttinger liquids

When the electrons are restricted into 1D spatial space, their physical properties are tremendously changed. They are in the so-called Luttinger liquids¹⁸, which are totally different to the Landau Fermi liquids.

Because of the special point structure of the Fermi surface of the 1D electrons, the low-energy quasiparticles are instable to interactions. Meanwhile, the low-energy particle-hole collective excitations become stable because there are no channels for them to decay into the quasiparticles. Thus the most stable low-energy excitations of the 1D Luttinger liquids are SDW and CDW excitations. The ground state of the 1D Luttinger liquids upon which the SDW and CDW excitations emerge is not adiabatically connected to the Landau Fermi liquids^{9,18}.

In the Luttinger model which has only linear dispersions and forward interactions, there are universal properties which are different to the Landau Fermi liquids⁹: (i) absence of the low-energy quasiparticles; (ii) the anomalous dimensions of the Fermi operators and the non-universal power-law correlations; (iii) spin-charge separation; (iv) the universal relations among the non-universal exponents of the correlation functions. These novel universal properties in solvable Luttinger model are conserved in the Luttinger liquids where non-linear dispersions and backward and Umklapp scattering interactions are involved. In language of the renormalization group, the Luttinger model is a low-energy fixed point of the 1D Luttinger liquids. Mathematically these universal properties of the Luttinger liquids are deeply rooted in the emergent conformal symmetry when the quantum electrons are restricted into 1D spatial space⁹.

C. Impurity-induced Anderson localization

When impurities are introduced into the electron system, the spatial translational symmetry is broken and the electronic Anderson localization can emerge^{39,40}.

It is shown from the renormalization group study⁴¹ that the Anderson localization is very universal. All the electronic states in 1D spatial space are localized. In 2D spatial space all the electronic states are also localized but the localization is marginal. In 3D spatial space things become more complex. In this case, part electronic states are localized while the others are extended, two of which are separated by the Mott mobility edge E_c ⁴². When the Fermi energy E_F is smaller than the mobility edge, i.e. $E_F < E_c$, the electrons are in a metal phase, and when $E_F > E_c$, the electrons are in an insulator phase. When $E_F = E_c$, the electrons are at a critical point which associates with the so-called Anderson metal-insulator phase transition.

After Anderson's seminal work in 1958³⁹, much efforts have been made on the microscopic mechanisms. One scenario is known as *weak* localization^{43,44} which states that the electron localization can come from the quantum interference of the backscattering processes, where one path and its time-reversal partner are constructive interference. This quantum interference can enhance backscattering probability and thus can lead to electron localization. In *weak* localization, the time-reversal symmetry should be conserved. However, it had been found by Anderson *et al.* that the electrons can be localized in a fractal Cayley-tree lattice where no loops are involved, thus the closed loops are not necessary for the electron localization⁴⁵.

The Anderson localization is still one interesting and elusive subject, and many new ideas and concepts such as multifractality and quantum nonergodicity are introduced in the modern study to understand the origin of the Anderson localization⁴⁶⁻⁴⁸.

IV. RESTRICTION ON INTERNAL DEGREES OF FREEDOM

In this section, we will consider the restriction on the fiber internal degrees of freedom of the quantum electrons. The following physics will be discussed: (i) quantum Hall effects of 2D electrons in strong magnetic field; (ii) spontaneous symmetry breaking with Fermi-surface instabilities which can lead to an ordered SDW, CDW, Pomeranchuk nematicity and SC state; (iii) the Mott-Hubbard physics with constraint on the local double occupation.

A. Strong magnetic field in 2D: Quantum Hall effects

The quantum Hall effects are one example of a topologically nontrivial state of matter. When electrons are confined in 2D spatial space and subjected to a strong magnetic field perpendicular to the plane, quantized Hall conductance plateaus as $\sigma_H = \nu e^2/h$ can be observed at low enough temperatures⁴⁹. The prefactor ν is known as *filling factor*, which is defined as a ratio between the electron density n and the density of magnetic flux: $\nu = n\Phi_0/B$, where B is the magnetic field and $\Phi_0 = hc/e$ is the flux quantum. The filling factor can take on either integers ($\nu = 1, 2, 3, \dots$) or some fractional values such as $\nu = 1/3, 2/5, 5/2, \dots$, etc. The former is known as integer quantum Hall effect (IQHE), which was first reported in 1980⁵⁰; and the later, known as fractional quantum Hall effect (FQHE), was first observed in 1982 for $\nu = 1/3$ ⁵¹, when the 2D electrons are confined at very high-quality semiconductor interfaces.

The IQHE can be understood from formation of the discrete Landau levels when electrons are confined in 2D with an external magnetic field perpendicular to the plane. For free electrons, the Landau levels are quantized, equally separated by a *cyclotron energy* gap ($\hbar\omega_c = \hbar eB/mc$), with each level being macroscopically degenerate. The filling factor ν , when it is an integer, simply counts the number of the filled Landau levels. It becomes clear that the Landau level can be characterized by a kind of topological invariant known as Chern number. Due to the 2D restriction on the movements of the electrons and the existence of the homogeneous magnetic field which explicitly breaks the time-reversal symmetry, the Chern numbers of the Landau levels are nontrivial. The summation of the Chern numbers of the filled Landau levels turns out to be the Hall conductance in fundamental units^{25,26}. From this point of view, the appearance of the Hall conductance plateaus is closely related to the discreteness of the topological number, i.e., the quantization of the Hall conductance is protected by a topological invariant.

Different from the IQHE which can be understood basically from a picture of the filled Landau levels with non-interacting electrons, for the FQHE, electron-electron interactions have to be taken into account seriously. It is worth to point out that the FQHE can be regarded as an extreme case of strongly interacting many-body systems. For a given partially filled Landau level, the Hamiltonian of the system contains only the interaction terms since all the electrons in this level have the same kinetic energy (ignoring the Landau level mixing and the spin degree of freedom for simplicity):

$$H = \mathcal{P} \sum_{j < k} \frac{e^2}{|\mathbf{r}_j - \mathbf{r}_k|} \mathcal{P}, \quad (7)$$

where \mathcal{P} is a Landau-level projector for the limited Hilbert space of the given Landau level. In this situation, there is no small parameter available and also no

normal state to serve as a starting point for traditional perturbative calculations. Moreover, the introduction of the projector \mathcal{P} makes the problem highly nontrivial.

Fortunately, starting from the seminar work of Laughlin¹⁹, much progress has been made to understand the nature of the FQHE. Among them, the concept of composite fermion, which was first proposed by Jain in 1989²⁰, provides a clear physical picture for it. A composite fermion is the bound state of an electron and an even number of quantized vortices. The vortices can cancel part of the Aharonov-Bohm phase originating from the external magnetic field, which in turn greatly reduces the strength of the effective magnetic field that the composite fermions experience. To some extent, the composite fermions can be regarded as *quasiparticles* of the FQHE. It is proper to stress here that due to the binding vortices, these composite particles are collective, nonlocal and topological quantum particles. The topological nature of the quantum Hall fluids can also be summarized into an effective field theory with the Chern-Simons term included⁵². With the help of the composite fermions, we can map most of the *fractional* Hall states of the electrons to the *integer* Hall states of the composite fermions, and various important features of the FQHE can be understood in this frame quantitatively with the help of numerical simulations⁵³.

B. Spontaneous symmetry breaking (I): SDW, CDW and Pomeranchuk instabilities

If the Fermi surface of a Landau Fermi liquid has nesting structure, the Fermi surface is unstable. Define the SDW operator $\mathbf{S}_{\mathbf{q}} = \sum_{\mathbf{k}} c_{\mathbf{k}+\mathbf{q}\sigma}^\dagger \frac{\sigma_{\alpha\alpha'}}{2} c_{\mathbf{k}\sigma'} = \sum_i \mathbf{S}_i e^{i\mathbf{q} \cdot \mathbf{r}_i}$ and the spin magnetic susceptibility $\chi_s(\mathbf{q}, \tau) = -\langle T_\tau \mathbf{S}_{\mathbf{q}}(\tau) \cdot \mathbf{S}_{-\mathbf{q}}(0) \rangle$. The nesting Fermi surface leads to a divergence of the static spin susceptibility, i.e., $\chi_s(\mathbf{Q}, \omega)|_{\omega=0} \rightarrow +\infty$, where the momentum \mathbf{Q} is the characteristic nesting momentum. This can lead to an instability of the Fermi surface in the spin channel and thus an ordered SDW state with breakdown of the Landau Fermi liquid.

When the Fermi surface has nesting feature, the Fermi-surface instability can also occur in charge channel. Define the CDW operator $n_{\mathbf{q}} = \sum_{\mathbf{k}\sigma} c_{\mathbf{k}+\mathbf{q}\sigma}^\dagger c_{\mathbf{k}\sigma} = \sum_i n_i e^{i\mathbf{q} \cdot \mathbf{r}_i}$ and the charge density susceptibility $\chi_c(\mathbf{q}, \tau) = -\langle T_\tau n_{\mathbf{q}}(\tau) \cdot n_{-\mathbf{q}}(0) \rangle$. The nesting Fermi surface can also lead to a divergence of the static charge susceptibility, $\chi_c(\mathbf{Q}, \omega)|_{\omega=0} \rightarrow +\infty$. Thus, the unstable Landau Fermi-liquid ground state can also be transited into an ordered CDW state. Which one of the SDW or the CDW state is the most stable ground state is determined by which susceptibility is more divergent. If the spin susceptibility is more divergent than the charge one, the SDW state is the more stable ground state.

In an ordered SDW or CDW state, the Fermi surface is gapped and the quasiparticles obtain finite gap. The single-particle Green's function has double singular poles

as a feature of the breakdown of the Landau Fermi liquids, as we have point out in Sec. II A.

On the perspective of the spontaneous symmetry breaking, the ordered SDW state breaks the spatial translational symmetry and the SU(2) spin rotational symmetry. In the ordered CDW state, only the spatial translational symmetry is broken. Both the SDW and the CDW instability are in particle-hole channel without particle number fluctuations ($\delta N = 0$). They are both correlated with the nesting symmetry of the Fermi surface.

There are another Fermi-surface instability, Pomeranchuk instability⁵⁴. If the Landau interaction in one charge channel has the form

$$H_I = U_\rho \sum_{\mathbf{k}\mathbf{k}'} \phi_{\mathbf{k}} \phi_{\mathbf{k}'} \bar{n}_{\mathbf{k}} \bar{n}_{\mathbf{k}'}, \quad (8)$$

where $\bar{n}_{\mathbf{k}} = \sum_{\sigma} c_{\mathbf{k}\sigma}^{\dagger} c_{\mathbf{k}\sigma}$. When U_ρ is negative and its absolute value is larger than a critical value $U_\rho^{(c)}$, there will be a Pomeranchuk deformation of the Fermi surface with the deformation symmetry determined by $\phi_{\mathbf{k}}$. Similarly, if the Landau interaction in one spin channel has the form

$$H_I = U_s \sum_{\mathbf{k}\mathbf{k}'} \phi_{\mathbf{k}} \phi_{\mathbf{k}'} \bar{\mathbf{S}}_{\mathbf{k}} \cdot \bar{\mathbf{S}}_{\mathbf{k}'}, \quad (9)$$

where $\bar{\mathbf{S}}_{\mathbf{k}} = \sum_{\sigma\sigma'} c_{\mathbf{k}\sigma}^{\dagger} \frac{\sigma_{\sigma\sigma'}}{2} c_{\mathbf{k}\sigma'}$. When U_s is negative and the absolute value is larger than a critical value $U_s^{(c)}$, a spin-dependent Pomeranchuk Fermi-surface deformation may occur. Compared to the SDW or the CDW Fermi-surface instability with finite momentum \mathbf{Q} , the Pomeranchuk Fermi-surface deformation can be regarded as a zero-momentum instability.

The Pomeranchuk Fermi-surface deformation is one manifestation of the electronic nematicity which breaks the rotational symmetry⁵⁵. The Pomeranchuk-Landau interactions (8) and (9) can lead to a phase transition with spontaneous rotational symmetry breaking and thus can result in a nematic state. The emergence of the Pomeranchuk nematic state is driven by energy gain from the Pomeranchuk-Landau interactions.

Contrary to the ordered SDW or CDW state with a finite gap at the Fermi surface, there is no gap opening in the nematic state. The analyticity of the single-particle Green's function does not change in the Pomeranchuk nematic phase transition. There is an adiabatic and analytic connection for the single-particle Green's function from the normal Landau Fermi liquid to the Pomeranchuk nematic state. Moreover, both the nematic and the normal state with or without Pomeranchuk Fermi-surface deformation can be described by the Landau Fermi-liquid theory, although there is a continuous rotational symmetry breaking.

C. Spontaneous symmetry breaking (II): Superconductivity

The Landau Fermi liquids have charge U(1) symmetry with invariant transformation $c_{i\sigma} \rightarrow c_{i\sigma} e^{i\theta}$. Superconductivity emerges when the global phase symmetry is broken. In a SC state, the SC order parameter

$$\hat{\Delta} = \sum_{\mathbf{k}\sigma\sigma'} \Delta_{\sigma\sigma'}(\mathbf{k}) c_{\mathbf{k}\sigma}^{\dagger} c_{-\mathbf{k}\sigma'}^{\dagger} \quad (10)$$

has finite average value. Without a spin-orbit coupling, we can decouple the spin singlet and the triplet components. In a spin singlet SC state, $\Delta_{\sigma\sigma'}(\mathbf{k}) = [-i\Lambda(\mathbf{k})\sigma_2]_{\sigma\sigma'}$, with $\Lambda(-\mathbf{k}) = \Lambda(\mathbf{k})$, and in a spin triplet state, $\Delta_{\sigma\sigma'}(\mathbf{k}) = [-i(\mathbf{d}(\mathbf{k}) \cdot \boldsymbol{\sigma})\sigma_2]_{\sigma\sigma'}$, with $\mathbf{d}(-\mathbf{k}) = -\mathbf{d}(\mathbf{k})$ ⁵⁶.

The finite average value $\langle \hat{\Delta} \rangle$ manifests macroscopic condensation of Cooper pairs. This macroscopic condensation stems from a fixed phase for all Cooper pairs which contribute to the whole condensate coherently. Because of the fixed phase in the SC condensate, the electron number fluctuates since the electron number \hat{N} and the phase θ are conjugate variables. The SC state arises in a phase transition when the charge U(1) symmetry is spontaneously broken. The phase physics of the SC condensate are sensitive to the electromagnetic field and have been shown by dramatic Josephson effects.

Upon a SC ground state, there are single-particle excitations with finite gap. The single-particle Green's function has double-pole singularity. The collective SDW and CDW excitations are also gapped. There are other collective excitations associated with the macroscopic condensate, the phase excitations and the amplitude Higgs modes. The longitudinal phase modes coupled with the electron density fluctuations become the Plasmon modes⁵⁷. The vortices are one transverse and topological phase excitations. The Higgs modes of the SC condensate^{58,59} are still in search in experiments.

The SC state can also be regarded as a low-energy Fermi-surface *nesting* physics in the particle-particle channel with a nesting momentum $\mathbf{Q} = 0$ (the momentum of the center of mass of the condensate). It has been shown by a renormalization group study that the SC instability is relevant upon the Landau Fermi liquids, and the SC state is one stable fixed point of the electron liquids⁶.

D. Gutzwiller projection: Mott-Hubbard physics

When the d- or f-electron shells of ions are not fully filled, the strong residual on-site Coulomb interactions, known as the Mott-Hubbard interactions, reduce largely the on-site double occupations. The relevant physics are generally described by the Hubbard model which involves both the kinetic hopping of the electrons and their on-site Mott-Hubbard interactions. The strong suppression

of local on-site double occupations can be described by the Gutzwiller projection⁶⁰

$$\mathcal{P} = \prod_i (1 - \alpha n_{i\uparrow} n_{i\downarrow}). \quad (11)$$

Generally $0 \leq \alpha \leq 1$ with $\alpha = 0$ for no any reduction of double occupations and $\alpha = 1$ for exact exclusion of any double occupation. The Gutzwiller projection is a restriction on the local basis states of the Hilbert space or the fiber-bundle space, where the local double-occupation basis states are reduced or removed.

The Gutzwiller projection restriction can lead to a Mott-Hubbard metal-insulator phase transition when the d- or f-electron shells are half-filled. When the virtual hopping processes are included, there is an effective low-energy t-J model with superexchange antiferromagnetic interactions. Then the Mott-Hubbard insulator state can show long-range antiferromagnetic order in quasi-2D or 3D compounds at low temperature. When additional electrons or holes are introduced into the half-filled Hubbard or t-J model, the long-range antiferromagnetic order is gradually destroyed to evolve into a spin-liquid or spin-glass state. When the doped electrons or holes become coherent in motion, superconductivity can emerge.

Anderson proposed a scenario of resonating valence bond (RVB) for the superconductivity in the hole-doped high-Tc cuprate superconductors⁶¹, where the Gutzwiller projection restriction plays driven roles for their low-energy physics. The superconductivity emerges when the holes are in coherent motion in a RVB spin-liquid background. This is a novel non-BCS mechanism for superconductivity.

Anderson's RVB proposal has been extensively studied by analytical and numerical methods. To deal with the local Gutzwiller projection, a U(1) gauge theory¹⁵ is introduced with the electron operator $c_{i\sigma}^\dagger$ re-formulated as

$$c_{i\sigma}^\dagger = f_{i\sigma}^\dagger b_i, \quad (12)$$

where the spin and the charge degree of freedom of the electron are separately carried by the spinon $f_{i\sigma}$ and the holon b_i , respectively. This decoupling is based upon an assumption that the Landau's quasiparticles are not well-defined in the Mott-Hubbard physics but with spin-charge separation. There is an emergent U(1) gauge symmetry with an additional gauge phase degree of freedom, which leads to strong correlations between the spinons and the holons. Novel physics such as non-BCS superconductivity, pseudogap, strange metal, etc. can arise. There are other various gauge theories for the Gutzwiller projection constraint with similar assumptions and similar outputs⁷⁷.

The Mott-Hubbard physics have also been extensively studied by the wavefunction Gutzwiller projection methods^{62,63}. Define the Gutzwiller projected state as $|\Psi\rangle_P = \mathcal{P}|\Phi_0\rangle$, where $|\Phi_0\rangle$ is a non-interacting Fermi gas or a mean-field BCS SC state. The Gutzwiller projected

Fermi gas is a renormalized Fermi liquid⁶², and the Gutzwiller projected BCS state is a renormalized BCS state⁶³⁻⁶⁵, both of which have well-defined quasiparticles with finite coherent weight. These results are in contrast to the original Anderson's RVB scenario where the electrons are in non-Fermi liquid state with spin-charge separation. Can the coherent quasiparticles survive the strong Mott-Hubbard interactions and how the unconventional superconductivity emerges from the normal pseudogap state or the strange metallic state are still elusive and are one big puzzle in the modern field of condensed matter.

The central role of the Mott-Hubbard interactions as reduction of on-site double occupations is a restriction on the local basis states in the Hilbert space or in the fiber-bundle space. The reduction of on-site double occupations obviously leads to breakdown of the Pauli exclusion principle and the Fermi-Dirac statistics, which would result in unusual correlations between the spin and the charge degree of freedom and breakdown of the Landau's Fermi-liquid theory.

V. RESTRICTION ON THE ELECTRONS FROM EXTERNAL DEGREES OF FREEDOM

In the Landau Fermi-liquid theory, only the low-energy forward-scattering interactions involve. Shankar has shown in his renormalization group study that these forward-scattering interactions are marginal and the Landau Fermi liquid is stable⁶.

In this section, we will consider the electrons which are coupled with external degrees of freedom. The models we will consider include the electrons coupled with Goldstone or gauge bosons, Anderson's hidden Fermi liquid, the electrons coupled with novel unparticles³⁴, the Kondo-impurity model and the Kondo-lattice model where the electrons are coupled with local moments.

A. Electrons coupled with Goldstone or gauge bosons: Watanabe-Vishwanath theorem

Recently, Watanabe and Vishwanath showed a theorem for stability of the Fermi-liquid behaviors and the Goldstone bosons in metal with spontaneous symmetry breaking⁶⁶. For a phase transition with continuous spontaneous symmetry breaking, define the generators $\{Q_a\}$ which are broken and associated with Goldstone modes. Consider the commutation relations

$$[Q_a, p_i] = i\Lambda_{ai}, \quad (13)$$

where p_i is the i -th component of the momentum \mathbf{p} . If $\Lambda_{ai} = 0$, the coupling between the electrons and the Goldstone modes vanishes in the limit of small energy-momentum transfer, and if $\Lambda_{ai} \neq 0$, the electron-Goldstone-boson coupling does not vanish. The vanishing coupling will conserve the stability of the electronic

Fermi-liquid states and the Goldstone modes; the non-vanishing coupling may lead to novel non-Fermi liquids and breakdown of the Goldstone modes. The Watanabe-Wishwanath theorem shows restriction effects on the metallic electrons from the external collective modes in the channel associated with spontaneously symmetry breaking.

The limit coupling between the electrons and the phonons in crystal compounds vanishes, and both the low-energy Fermi-liquid states and the phonons are stable. Similarly in ferromagnetic metal, the electron-magnon coupling vanishes and the magnons are stable. Generally, all of the Goldstone modes which are associated with any internal symmetry breaking are stable.

The non-vanishing coupling between the electrons and the nematic modes which comes from spontaneous breaking of the rotational symmetry may lead to non-Fermi liquids and overdamped Goldstone modes. The non-Fermi liquids with rotational symmetry breaking are extensively studied^{55,66,67}. The coupling between the electrons and the transverse gauge modes also does not vanish, which can lead to novel non-Fermi liquid behaviors at extremely low temperature^{66,68}.

B. Anderson's Hidden Fermi liquid

Recently, Anderson proposed a Hidden Fermi-liquid theory for the high-Tc cuprate superconductors^{33,69}. For the projected t-J model with Hamiltonian H_{tJ} , there is one to one correspondence between the exact eigenstates $|\Psi\rangle$ and the unprojected states $|\Phi\rangle$, i.e.,

$$|\Psi\rangle = \mathcal{P}|\Phi\rangle, \quad (14)$$

where \mathcal{P} is the Gutzwiller projection operator with exact remove of the double occupations defined in (11). It is shown that if $H_{tJ}|\Psi\rangle = E|\Psi\rangle$, then $H_{tJ}|\Phi\rangle = E|\Phi\rangle$. $|\Phi\rangle$ can be non-interacting Fermi-gas states or mean-field BCS states.

Since $\mathcal{P}c_{i\sigma}^\dagger \mathcal{P} = c_{i\sigma}^\dagger(1 - n_{i\bar{\sigma}})$, the single-particle Green's function at ground state $G_{ij}(t) = -i\langle\Psi_G|T_t c_{i\sigma}(t)c_{j\sigma}^\dagger(0)|\Psi_G\rangle$ can be simplified as

$$G_{ij}(t) \sim G_{ij}^{(0)}(t)G_{ij}^{(*)}(t), \quad (15)$$

where $G_{ij}^{(0)}(t) = -i\langle\Phi_G|c_{i\sigma}(t)c_{j\sigma}^\dagger(0)|\Phi_G\rangle$ describes the electron free propagation, and $G_{ij}^{(*)}(t) = \langle\Phi_G|T_t(1 - n_{i\bar{\sigma}})(t)(1 - n_{j\bar{\sigma}})(0)|\Phi_G\rangle$ describes the background scatterings on the electron propagation. $|\Psi_G\rangle$ and $|\Phi_G\rangle$ are the ground states associated with H_{tJ} by (14).

With this simplification, Anderson showed that³³

$$G(\mathbf{k}, \omega) \sim \frac{1}{(\omega - v_F k)^{1-a}}, \quad (16)$$

where v_F is the Fermi velocity. The finite exponent a ($a \neq 0$) stems from the background scatterings in $G^{(*)}$.

The finite exponent a has dramatic effects, since it shows a branch-cut singularity in the single-particle Green's function $G(\mathbf{k}, \omega)$ and thus non-Fermi liquid behaviors. Physically and mathematically, the form of the Green's function (16) comes from the convolution of the bare electron propagation and the background fluctuations. Here the electrons are restricted by the Gutzwiller projection of remove of the double occupations, which is related to the underlying gauge symmetries in the RVB scenario and breakdown of the Fermi-Dirac statistics as we have discussed in Section IV D.

The scenario of the Hidden Fermi liquid for the high-Tc cuprate superconductors can be extended into the 2D electrons with quantum Hall effects⁷⁰.

C. Electrons coupled with unparticles

Limtragoon *et al.* introduced a phenomenological model to describe the unconventional normal states of the high-Tc cuprate superconductors³⁴. In the proposed model, the electrons are coupled with the scalar Bose unparticles which have novel propagator

$$\mathcal{D}_\mu(\mathbf{q}, \nu_n) = \frac{1}{(\nu_n^2 + E_\mathbf{q}^2)^{1-\alpha}}, \quad (17)$$

where the exponent $\alpha = 1 - \frac{d+1}{2} + d_\mu$ with d the spatial dimension and d_μ the scaling dimension of the unparticle field. The Green's function of the unparticles \mathcal{D}_μ has branch-cut singularity, rather than the general pole singularity, thus they are not well-defined Bosons. It can be shown that at low temperature and low energy, the self-energy of the electrons shows non-Fermi liquid behaviors,

$$\Im\Sigma \sim T^{d-2+2\alpha}, |\omega|^{d-2+2\alpha}. \quad (18)$$

The roles of the Bose unparticles are to introduce restriction on the electrons in one special scalar channel. The unparticle propagator with branch-cut singularity plays similar roles in the non-Fermi liquid behaviors to the power-law Coulomb interaction in fractal space³².

D. Conduction electrons coupled with local moments: Kondo effects and heavy fermions

When the conduction electrons are coupled with local magnetic impurities or local d or f electrons, novel physics emerge, such as the universal Kondo effects and the heavy fermions with unconventional superconductivity.

When magnetic impurities are introduced into a metallic conductor, there are some universal Kondo effects observed⁷¹: (i) a resistivity minimum; (ii) universality in the specific heat $C_V \sim \frac{1}{T} f_c\left(\frac{T}{T_K}\right)$; (iii) universality in the resistivity $\rho(T) \sim f_\rho\left(\frac{T}{T_K}\right)$. The Kondo temperature T_K is the unique characteristic energy scale for the low-energy universal Kondo effects.

The above universal Kondo effects can be described by the Kondo-impurity model with the Hamiltonian

$$H = \sum_{\mathbf{k}\sigma} \varepsilon_{\mathbf{k}} c_{\mathbf{k}\sigma}^\dagger c_{\mathbf{k}\sigma} + J \sum_{\sigma\sigma'} c_{i_0\sigma}^\dagger \frac{\boldsymbol{\sigma}_{\sigma\sigma'}}{2} c_{i_0\sigma'} \cdot \mathbf{S}_f, \quad (19)$$

where $\{c, c^\dagger\}$ are operators for the conduction electrons and \mathbf{S}_f is the spin of the impurity which locate at site i_0 . The Kondo effects are deeply rooted in the Kondo resonance which stems from local coupling of the conduction electrons and the magnetic impurity. It should be noted that the Kondo coupling J has a dramatic character, an asymptotically free behavior, i.e., its renormalized value is small at high energy but is large at low energy⁷².

When the magnetic moments are in a periodical crystal configuration as in most heavy fermion superconductors, one proper model is the Kondo-lattice model with the Hamiltonian

$$H = \sum_{\mathbf{k}\sigma} \varepsilon_{\mathbf{k}} c_{\mathbf{k}\sigma}^\dagger c_{\mathbf{k}\sigma} + J \sum_{i\sigma\sigma'} c_{i\sigma}^\dagger \frac{\boldsymbol{\sigma}_{\sigma\sigma'}}{2} c_{i\sigma'} \cdot \mathbf{S}_{f,i}. \quad (20)$$

One special unconventional physics in the heavy fermion superconductors is the emergence of the heavy fermions which have Landau Fermi-liquid behaviors but with huge renormalized mass (m^*/m can reach 1000). Physically, these heavy fermions are analogue of the Kondo resonance and they form into a nearly flat band near the Fermi energy.

Both the Kondo resonance and the heavy fermions can be regarded as one composite fermions which come from the binding of the conduction electrons and the spin-flip of the local magnetic moments⁷¹. Mathematically, the composite fermions can be defined by

$$\sum_{\sigma'} (\boldsymbol{\sigma} \cdot \mathbf{S}_{f,i}(t))_{\sigma\sigma'} c_{i\sigma'}(t') \rightarrow \Delta(t-t') f_{i\sigma}(t'), \quad (21)$$

where $f_{i\sigma}$ is an operator for the composite fermions and $\Delta(t-t')$ is analog of the Cooper pair wavefunction. These composite fermions are local in spatial space but are strongly correlated in dynamic processes⁷³, thus much non-local in temporal space. They form *local* Fermi liquids⁷⁴. In this scenario, the Kondo resonance and the heavy fermions are emergent composite fermions which come from the interaction-driven restriction on temporal correlations between the itinerant and the localized electrons in spin channel.

To study the unconventional physics in the heavy fermion superconductors is still one challenge in the modern field of condensed matter, where the quantum phase transitions and the quantum criticality are examples of the main focus^{8,11,75}.

VI. DISCUSSIONS AND PROPOSALS

In the last three decades, many strongly correlated electronic systems are discovered in experiments which

display unconventional properties beyond the Landau Fermi-liquid theory. One major challenge in the modern condensed matter field is to construct theories to understand these novel phenomena and to make reliable predictions on the properties of the systems.

In this article, we present a new perspective for the unconventional electrons which are beyond the Landau Fermi-liquid theory. Our point of view is that most of the unconventional electronic physics stem from restriction on some degrees of freedom of the electrons, which may involve the spatial position, the momentum, the time variable, the spin, the charge, the phase, and the basis states, etc. For quantum electrons, the degrees of freedom are related either to the symmetries or to the fiber-bundle topology of the quantum wavefunction or the quantum field. Therefore, the restriction on the degrees of freedom can be applied through (i) reduction of the symmetries, (ii) emergence of new symmetries, or (iii) emergence of the nontrivial topology of the fiber-bundle space.

Restriction on the electrons for novel electronic physics which are beyond the Landau Fermi-liquid theory is summarized in Table (II). The anomalous dimensions in critical phenomena shows their fractal characters, which are deeply rooted in an emergent scaling symmetry and a non-linear coupling of the high- and low-energy modes. The Luttinger liquids are in a stable fixed point with an emergent conformal symmetry in the reduced 1D spatial space. The Anderson localization comes from impurity-induced randomness in spatial space. The quantum Hall effects stem from the confinement of the electrons into 2D spatial space and under strong magnetic field. Non-trivial topology and fractional statistics emerge, the latter of which is in the fractional quantum Hall effects. The spontaneous symmetry breaking in the spin or the charge channel can lead to SDW, CDW, or Pomeranchuk nematicity with Fermi-surface instability. The superconductivity as macroscopic condensation of Cooper pairs comes from the fixed phase when the charge U(1) symmetry is broken. The Mott-Hubbard physics are dominated by suppression of the on-site double occupations, which results in emergent gauge symmetries and breakdown of the Fermi-Dirac statistics. The unconventional electronic physics can also come from restriction on the electrons from external degrees of freedom, such as Goldstone or gauge modes in the symmetry broken states, the background scatterings in Anderson's hidden Fermi liquid, unparticles with anomalous dimension, local magnetic moments in Kondo effects and heavy fermion superconductors.

Following our perspective that the unconventional electronic physics stem from restriction on some degrees of freedom of the electrons, there can be two steps in order to establish the theory for the unconventional physics which we are interested in: first is to identify the degrees of freedom which are restricted, e.g., the relevant degrees of freedom can be in the base spatial-temporal space or the fiber internal space; secondly we study how to take

TABLE II: Restriction for the novel electronic physics with breakdown of the Landau Fermi liquids. Denotations and abbreviation are defined as following: \mathbf{r}, \mathbf{k} - spatial degree(s) of freedom (DOF), t - temporal DOF, S - spin, C - charge, spc. - space, dim. - dimension, B/F - base/fiber space, sym. - symmetry, emgt. - emergent, brk. - breaking, nos. - numbers, momt. - momentum, stats. - statistics, chanl. - channel, ST - spatial translational, TR - time-reversal, FS - Fermi-surface, FD - Fermi-Dirac. The “randomness stats.” for Anderson localization denotes the new concepts of the multifractality⁴⁶ and nonergodicity^{47,48} in the modern study.

novel physics	DOF	B/F spc.	quantum nos.	symmetry or topology
critical phenomena	\mathbf{r}	B	\mathbf{k}	emgt. scaling sym.
1D luttinger liquids	\mathbf{r}	B	\mathbf{k}	emgt. conformal sym.
Anderson localization	\mathbf{r}	B		ST sym. brk. (randomness stats.)
IQHE	\mathbf{r} , Landau levels	B & F	Chern nos.	TR sym. brk., emgt. nontrivial topology
FQHE	\mathbf{r} , Landau levels	B & F	Chern nos.	TR sym. brk., emgt. nontrivial topology, emgt. fractional stats.
SDW	FS, S	B & F	\mathbf{k}, σ	ST and spin SU(2) sym. brk. (FS nesting)
CDW	FS, C	B & F	\mathbf{k}, n	ST sym. brk. (FS nesting)
Pomeranchuk	FS, C/S	B & F	angular momt., n/σ	rotational sym. brk.
SC	phase θ	F	n	charge U(1) sym. brk.
Mott-Hubbard physics	S, C , local basis	F	σ, n	emgt. gauge sym., FD stats. brk.
electrons with Goldstone modes	of sym. brk.	B or F		sym. brk.
hidden Fermi liquid	S, C	F		emgt. gauge sym. (underlying), FD stats. brk.
electrons with unparticles	of scalar chanl.	F		topology with an effective fractal dim.
electrons with local moments	t, S	B & F	ω, σ	emgt. FD Stats. of composite fermions

into account the restriction in the microscopic processes, such as from external constraints or in interaction-driven processes. At present the interaction-driven restrictions appear in most electronic non-Fermi liquids, which however are difficult to analyze in microscopic level. One possible way to avoid the difficulty from the many-body interactions is to introduce artificial external potentials at the beginning to simulate the roles of the interactions.

Our perspective of restriction on the degrees of freedom for novel electronic physics can also be a guideline for the innovations of the electronic devices with novel physical properties. We can control manually the electrons for artificial electronic devices by operation upon any of the degrees of freedom, such as the spatial position, the time

variable, the spin, the charge, the phase, the topology, etc. The electrons in special spatial configurations with nontrivial topology or with external artificial potentials may show interesting properties.

One special notice is that the temporal variable may be one special degree of freedom whole roles are not well studied. The non-local temporal correlations and other possible interaction-driven restriction on the temporal variable may be relevant to some non-Fermi liquid physics such as the quantum criticality.

Acknowledgement We thank Tao Li and Yin Zhong for helpful discussions. H.T.L. acknowledges the grant from the Natural Science Foundation of China (Grant No. 11474136).

* Electronic address: suyh@ytu.edu.cn

† Electronic address: hantao.lu@gmail.com

¹ J. P. Landau, JETP **30**, 1058 (1956).

² J. P. Landau, JETP **32**, 59 (1957).

³ J. P. Landau, JETP **35**, 97 (1958).

⁴ P. Coleman, *Introduction to Many Body Physics, Chapter 7* (Cambridge University Press, Cambridge, UK, 2015), 1st ed.

⁵ A. A. Abrikosov, L. P. Gor'kov, and I. Y. Dzyaloshinskii, *Quantum field theoretical methods in statistical physics*

(Perpamon Press Ltd., 1965), 2nd ed.

⁶ R. Shankar, Rev. Mod. Phys. **66**, 129 (1994).

⁷ C. M. Varma, Z. Nussinovb, and W. van Saarloos, Phys. Rep. **361**, 267 (2002).

⁸ G. R. Stewart, Rev. Mod. Phys. **73**, 797 (2001).

⁹ J. Voit, Rep. Prog. Phys. **57**, 977 (1994).

¹⁰ E. Abrahams, ed., *Lecture notes in 50 years of Anderson localization* (World Scientific, Singapore, 2010), 1st ed.

¹¹ P. Coleman, in *Lecture notes for Autumn School on Correlated Electrons: Many-Body Physics: From Kondo to Hub-*

- bard* (arXiv:1509.05769) (2015).
- ¹² B. Keimer, S. A. Kivelson, M. R. Norman, S. Uchida, and J. Zaanen, *Nature* (London) **518**, 179 (2015).
 - ¹³ M. R. Norman, *Novel Superfluids*, p. 23-79 (arXiv:1302.3176) (Oxford University Press, Oxford, 2014), 2nd ed.
 - ¹⁴ P. W. Anderson, P. A. Lee, M. Randeria, T. M. Rice, N. Trivedi, and F. C. Zhang, *J. Phys.: Condens. Matter* **16**, R755 (2004).
 - ¹⁵ P. A. Lee, N. Nagaosa, and X.-G. Wen, *Rev. Mod. Phys.* **78**, 17 (2006).
 - ¹⁶ X. H. Chen, P. C. Dai, D. L. Feng, T. Xiang, and F.-C. Zhang, *National Science Review* **1**, 371 (2014).
 - ¹⁷ G. R. Stewart, *Rev. Mod. Phys.* **83**, 1589 (2011).
 - ¹⁸ F. D. M. Haldane, *J. Phys. C: Solid State Phys.* **14**, 2585 (1981).
 - ¹⁹ R. B. Laughlin, *Phys. Rev. Lett.* **50**, 1395 (1983).
 - ²⁰ J. K. Jain, *Phys. Rev. Lett.* **63**, 199 (1989).
 - ²¹ L. D. Landau and E. M. Lifshitz, *Statistical Physics, Part 1*, Course of Theoretical Physics, Vol. 5 (Beijing World Publishing Corporation by arrangement with Betterworth-Heinemann, 1999), 3rd ed.
 - ²² K. G. Wilson and J. Kogut, *Physics Report* (Section C of *Physics Letters*) **12**, 75 (1974).
 - ²³ P. A. M. Dirac, *The Principles of Quantum Mechanics* (Science Press, 2008), 4th ed.
 - ²⁴ S. Weinberg, *Lectures on Quantum Mechanics* (Cambridge University Press, New York, 2013), 1st ed.
 - ²⁵ D. J. Thouless, M. Kohmoto, M. P. Nightingale, and M. den Nijs, *Phys. Rev. Lett.* **49**, 405 (1982).
 - ²⁶ J. E. Avron, D. Osadchy, and R. Seiler, *Physics Today* **56**, 38 (2003).
 - ²⁷ M. Z. Hasan and C. L. Kane, *Rev. Mod. Phys.* **82**, 3045 (2010).
 - ²⁸ X.-L. Qi and S.-C. Zhang, *Rev. Mod. Phys.* **83**, 1057 (2011).
 - ²⁹ C. M. Varma, P. B. Littlewood, S. Schmitt-Rink, E. Abrahams, and A. E. Ruckenstein, *Phys. Rev. Lett.* **63**, 1996 (1989).
 - ³⁰ A. V. Chubukov, D. Pines, and J. Schmalian, *The Physics of Conventional and Unconventional Superconductors* (arXiv:cond-mat/0201140) (Springer-Verlag, 2002), 1st ed.
 - ³¹ K.-Y. Yang, T. M. Rice, and F.-C. Zhang, *Phys. Rev. B* **73**, 174501 (2006).
 - ³² P.-A. Bares and X.-G. Wen, *Phys. Rev. B* **48**, 8636 (1993).
 - ³³ P. W. Anderson, *Phys. Rev. B* **78**, 174505 (2008).
 - ³⁴ K. Limtragoon, C. Setty, Z. Leong, and P. W. Phillips, arXiv:1608.06637 (2016).
 - ³⁵ S.-K. Ma, *Modern Theory of Critical Phenomena*, Advanced Book Program (Westview Press, 2000), 1st ed.
 - ³⁶ N. Goldenfeld, *Lectures on Phase Transitions and Renormalization Group*, The Advanced Book Program (Perseus Books, Reading, Massachusetts, 1992), 1st ed.
 - ³⁷ M. Suzuki, *Prog. Theor. Phys.* **69**, 65 (1983).
 - ³⁸ H. Kröger, *Phys. Rep.* **323**, 81 (2000).
 - ³⁹ P. W. Anderson, *Phys. Rev.* **109**, 1492 (1958).
 - ⁴⁰ A. Lagendijk, B. van Tiggelen, and D. S. Wiersma, *Physics Today* **62**, 24 (2009).
 - ⁴¹ E. Abrahams, P. W. Anderson, D. C. Licciardello, and T. V. Ramakrishnan, *Phys. Rev. Lett.* **42**, 673 (1979).
 - ⁴² N. Mott, *J. Phys. C: Solid State Phys.* **20**, 3075 (1987).
 - ⁴³ D. Vollhardt and P. Wölfle, *Phys. Rev. B* **22**, 4666 (1980).
 - ⁴⁴ D. Vollhardt and P. Wölfle, *Phys. Rev. Lett.* **48**, 699 (1982).
 - ⁴⁵ R. Abou-Chakra, P. W. Anderson, and D. J. Thouless, *J. Phys. C: Solid State Phys.* **6**, 1734 (1973).
 - ⁴⁶ F. Evers and A. D. Mirlin, *Rev. Mod. Phys.* **80**, 1355 (2008).
 - ⁴⁷ W. De Roeck, F. Huveneers, M. Müller, and M. Schiulaz, *Phys. Rev. B* **93**, 014203 (2016).
 - ⁴⁸ X. Li, J. H. Pixley, D.-L. Deng, S. Ganeshan, and S. Das Sarma, *Phys. Rev. B* **93**, 184204 (2016).
 - ⁴⁹ R. Prange, M. Cage, K. Klitzing, S. Girvin, A. Chang, F. Duncan, M. Haldane, R. Laughlin, A. Pruisken, and D. Thouless, *The Quantum Hall Effect*, Graduate Texts in Contemporary Physics (Springer New York, 1990), 2nd ed.
 - ⁵⁰ K. v. Klitzing, G. Dorda, and M. Pepper, *Phys. Rev. Lett.* **45**, 494 (1980).
 - ⁵¹ D. C. Tsui, H. L. Stormer, and A. C. Gossard, *Phys. Rev. Lett.* **48**, 1559 (1982).
 - ⁵² S. C. Zhang, T. H. Hansson, and S. Kivelson, *Phys. Rev. Lett.* **62**, 82 (1989).
 - ⁵³ J. K. Jain, *Composite Fermions* (Cambridge University Press, 2007), 1st ed.
 - ⁵⁴ I. I. Pomeranchuk, *JETP* **8**, 361 (1959).
 - ⁵⁵ E. Fradkin, S. A. Kivelson, M. J. Lawler, J. P. Eisenstein, and A. P. Mackenzie, *Annual Reviews of Condensed Matter Physics* **1**, 153 (2010).
 - ⁵⁶ M. Sigrist and K. Ueda, *Rev. Mod. Phys.* **63**, 239 (1991).
 - ⁵⁷ P. W. Anderson, *Phys. Rev.* **112**, 1900 (1958).
 - ⁵⁸ P. B. Littlewood and C. M. Varma, *Phys. Rev. Lett.* **47**, 811 (1981).
 - ⁵⁹ P. B. Littlewood and C. M. Varma, *Phys. Rev. B* **26**, 4883 (1982).
 - ⁶⁰ M. C. Gutzwiller, *Phys. Rev. Lett.* **10**, 159 (1963).
 - ⁶¹ P. W. Anderson, *Science* **235**, 1196 (1987).
 - ⁶² D. Vollhardt, *Rev. Mod. Phys.* **56**, 99 (1984).
 - ⁶³ B. Edegger, V. N. Nuthukumar, and C. Gros, *Advances in Physics* **56**, 927 (2007).
 - ⁶⁴ H. Y. Yang, F. Yang, Y. J. Jiang, and T. Li, *J. Phys. Condens. Matter* **19**, 016217 (2007).
 - ⁶⁵ S. Yunoki, *Phys. Rev. B* **74**, 180504 (2006).
 - ⁶⁶ H. Watanabe and A. Vishwanath, *Proc. Natl. Acad. Sci.* **111**, 16314 (2014).
 - ⁶⁷ S. Lederer, Y. Schattner, E. Berg, and S. A. Kivelson, *Phys. Rev. Lett.* **114**, 097001 (2015).
 - ⁶⁸ A. M. Tsvelik, *Quantum field theory in condensed matter physics* (Cambridge University Press, Cambridge, England, 1995), 1st ed.
 - ⁶⁹ P. W. Anderson and P. A. Casey, *Phys. Rev. B* **80**, 094508 (2009).
 - ⁷⁰ J. K. Jain and P. W. Anderson, *Proc. Natl. Acad. Sci.* **106**, 9131 (2009).
 - ⁷¹ P. Coleman, *Introduction to Many Body Physics, Chapter 17 and 18* (Cambridge University Press, Cambridge, UK, 2015), 1st ed.
 - ⁷² P. W. Anderson, *J. Phys. C: Solid State Phys.* **3**, 2436 (1970).
 - ⁷³ Q. Si, S. Rabello, K. Ingersent, and J. L. Smith, *Phys. Rev. B* **68**, 115103 (2003).
 - ⁷⁴ P. Nozières, *J. Low Temp. Phys.* **17**, 31 (1974).
 - ⁷⁵ H. v. Löhneysen, A. Rosch, M. Vojta, and P. Wölfle, *Rev. Mod. Phys.* **79**, 1015 (2007).
 - ⁷⁶ A fiber bundle E is a space which is locally homomorphic to a product space $B \times F$ and but globally may have topological structure. In formal mathematics, a fiber bundle is a structure (E, B, π, F) , where a continuous surjection is

defined as $\pi : E \rightarrow B$. For any $x \in E$, there is an open neighborhood $U \subset B$ such that $\pi^{-1}(U)$ is homomorphic to $U \times F$. The space B is called the base space, F the fiber and E the total space. From Wikipedia on “fiber bundle”, https://en.wikipedia.org/wiki/Fiber_bundle.

⁷⁷ The brilliant RVB proposal and the various gauge theories are still in doubt, as the proposed spinons and holons have not been conformed in experiments.